J. Appl. Maths Mechs Vol. 56, No. 5, pp. 707–723, 1992
Printed in Great Britain.

0021-8928/92 $24.00 + 0.00

BOUNDED CONTROLS IN DISTRIBUTED-PARAMETER SYSTEMS†

F. L. CHERNOUS'KO

Moscow

(*Received* 27 *February* 1992)

Control systems with distributed parameters, described by partial differential equations, solvable with respect to the first or second time derivative, are considered. The controls on the right-hand sides of the equations are assumed to be bounded in absolute magnitude. A control method is proposed which brings the controlled system into the null state in a finite time. The proposed approach is based on decomposing the system and applying the time-optimal control for each mode of motion obtained by Fourier-expanding the solution. Estimates for the duration of the control process are obtained. Sufficient conditions for the problem to be solvable are given. Examples are presented.

1. STATEMENT OF THE PROBLEM

CONTROL systems with distributed parameters described by linear partial differential equations are considered. We shall consider in tandem the equation

$$w_t = Aw + v \tag{1.1}$$

solved with respect to the first time derivative, and the equation

$$w_{tt} = Aw + v \tag{1.2}$$

solved with respect to the second derivative.

In Eqs (1.1) and (1.2) $w(x, t)$ is the scalar function of the n-dimensional spatial coordinate vector $x = (x_1, \ldots, x_n)$ and time t which describes the state of the system, v is the required control, and A is a linear differential operator containing partial derivatives with respect to the coordinates x_i, $i = 1, \ldots, n$. The coefficients of the operator A do not depend on t, and its order ord$A$ is assumed to be even and equal to $2m$.

The most important and frequently encountered examples of Eqs (1.1) and (1.2), which we shall have in mind in the following, are: (1) the heat-conduction equation, which is obtained from (1.1) if $m = 1$ and $A = \Delta$ is the Laplace operator; (2) the wave equation obtained from (1.2) with $m = 1$ and $A = \Delta$; (3) the equation for the vibrations of an elastic beam or plate, obtained from (1.2) with $m = 2$, $A = -\Delta^2$ and $n = 1, 2$, respectively. Equations (1.1) and (1.2) also describe heat-condition processes and vibrations in an inhomogeneous medium if

$$Aw = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial w}{\partial x_i}\right], \quad m = 1$$

where $a(x)$ is a specified function describing the inhomogeneity of the medium.

Equations (1.1), (1.2) are considered in some bounded domain of variation for the spatial variables $x \in \Omega$ and for $t \geqslant 0$. At the boundary Γ of the domain Ω a homogeneous boundary condition of the following form should be satisfied

† *Prikl. Mat. Mekh.* Vol. 56, No. 5, pp. 810–826, 1992.

$$Mw = 0, \quad M = (M_1, \ldots, M_m), \quad x \in \Gamma \tag{1.3}$$

Here M_j is a linear differential operator of order $\operatorname{ord} M_j < 2m$, $(j = 1, \ldots, m)$ with coefficients independent of t. In particular, for $m = 1$ the operator M is scalar and has the form

$$Mw = b_0(x)w + b_1(x)\partial w/\partial x$$

where $b_0(x)$ and $b_1(x)$ are functions given on Γ. Condition (1.3) can, in particular, become the Dirichlet condition (for $b_0 = 1$, $b_1 = 0$) or the Neumann condition (for $b_0 = 0$, $b_1 = 1$).

The initial conditions have the form

$$w(x, 0) = w_0(x), \quad x \in \Omega \tag{1.4}$$

for Eq. (1.1) and

$$w(x, 0) = w_0(x), \quad w_t(x, 0) = w_{t0}(x), \quad x \in \Omega \tag{1.5}$$

for Eq. (1.2).

The constraint

$$|v(x, t)| \leqslant v^0, \quad x \in \Omega, \quad t \geqslant 0 \tag{1.6}$$

is imposed on the control function v, where $v^0 > 0$ is a given constant.

We will now formulate the second problem.

It is required to construct a control $v(x, t)$ satisfying condition (1.6) and such that the corresponding solution of (1.1) or (1.2) with boundary condition (1.3) and the corresponding initial condition (1.4) or (1.5) vanishes in some finite (unspecified) time $T > 0$. More precisely, everywhere in Ω the condition $w(x, T) = 0$ should be satisfied for Eq. (1.1) and $w(x, T) = w_t(x, T) = 0$ should be satisfied for Eq. (1.2). Obviously, if one puts $v \equiv 0$ for $t \geqslant T$, the solution remains identically equal to zero for $t > T$.

The boundary of the domain Ω is assumed to be smooth; the examples also include cases with piecewise-smooth boundaries. Requirements on the initial functions, and the function classes to which the solutions of the problems belong in various cases, are considered in Sec. 9.

Many publications have been devoted to systems with distributed parameters, for example [1–6]. The control method proposed below differs from the earlier ones. It enables one to construct a constrained control in closed form and ensures that the system is brought into a given state in a finite time. This method uses a decomposition of the original system into simple subsystems and in this sense is close in spirit to [7], where systems with a finite number of degrees of freedom were considered.

2. DECOMPOSITION OF THE CONTROL PROBLEM

The solution of the problem is based on the Fourier method. To apply it we will first consider the following eigenvalue problem, corresponding to the initial-boundary-value problems (1.1)–(1.5) for $v = 0$.

The problem is to find function $\varphi(x)$, $x \in \Omega$ and corresponding constants λ that satisfy a linear homogeneous equation with boundary conditions

$$A\varphi = -\lambda\varphi, \quad x \in \Omega; \quad M\varphi = 0, \quad x \in \Gamma \tag{2.1}$$

It is known that under specified conditions (for self-conjugate elliptic equations and, in particular, for the Laplace equation, i.e. when $A = \Delta$), the eigenvalue problem (2.1) has the following properties.

There is a discrete denumerable spectrum of positive eigenvalues λ_k, which can be numbered in non-decreasing order: $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$, with $\lambda_k \to \infty$ as $k \to \infty$. In certain cases, for example, for the Laplace operator $A = \Delta$ with Neumann conditions, there is also a zero eigenvalue $\lambda_0 = 0$. That case will also be considered. To these eigenvalues there corresponds an orthogonal system of eigenfunc-

$\varphi_k(x)$, complete in the domain Ω. Normalizing these functions, we obtain an orthonormal system of functions $\varphi_k(x)$, possessing the following properties

$$\begin{aligned} &A\varphi_k = -\lambda_k\varphi_k, \quad x \in \Omega; \quad M\varphi_k = 0, \quad x \in \Gamma \\ &(\varphi_k, \varphi_i) = \int_\Omega \varphi_k(x)\varphi_i(x)dx = \delta_{ki} \end{aligned} \tag{2.2}$$

Here δ_{ki} is the Kronecker delta. The index k in (2.2) and below, unless otherwise stated, runs over values from 0 to ∞ when there is a zero eigenvalue and from 1 to ∞ when there is none. Summation over k will also be performed over the ranges given above.

We now use the Fourier method to separate the time (t) and space (x) dependence. Solutions of Eqs (1.1) and (1.2) will be sought in the form of eigenfunction expansions

$$w(x, t) = \Sigma q_k(t)\varphi_k(x) \tag{2.3}$$

Here the $q_k(t)$ are certain functions of time.

The control v in (1.1) and (1.2) is also represented in the form of an expansion

$$v(x, t) = \Sigma u_k(t)\varphi_k(x) \tag{2.4}$$

where the $u_k(t)$ are currently unknown functions of time.

Substituting expansions (2.3) and (2.4) into Eq. (1.1) we obtain

$$\Sigma \dot{q}_k\varphi_k = \Sigma(q_k A\varphi_k + u_k\varphi_k)$$

Here and below the dots denote time derivatives.

We use the equations $A\varphi_k = \lambda_k\varphi_k$ from (2.2) together with the orthogonality of the φ_k. As a result we have the system of equations

$$\dot{q}_k + \lambda_k q_k = u_k \tag{2.5}$$

Similarly, substituting expansions (2.3) and (2.4) into (1.2), we obtain

$$\ddot{q}_k + \omega_k^2 q_k = u_k \tag{2.6}$$

Here and below the ω_k are the frequencies of the natural modes, given by

$$\omega_k = \lambda_k^{1/2}, \quad 0 = \omega_0 \leqslant \omega_1 \leqslant \omega_2 \leqslant \ldots \tag{2.7}$$

We note that a solution of the form (2.3) satisfies, by construction, the boundary condition (1.3), because according to (2.2) all the eigenfunctions satisfy this condition.

We substitute solution (2.3) into the initial conditions (1.4) and (1.5) and use the orthonormality of the eigenfunctions (2.2). We obtain initial conditions for problem (2.5) in the form

$$q_k(0) = q_k^0 = \int_\Omega w_0(x)\varphi_k(x)dx \tag{2.8}$$

and for problem (2.6) in the form

$$\begin{aligned} &q_k(0) = q_k^0 = \int_\Omega w_0(x)\varphi_k(x)dx \\ &\dot{q}_k(0) = (\dot{q}_k)^0 = \int_\Omega w_{t0}(x)\varphi_k(x)dx \end{aligned} \tag{2.9}$$

The original control problem for the partial differential equations (1.1) and (1.2) has thus been reduced to a control problem for linear control systems of infinite order (2.5) and (2.6). On the control functions $u_k(x)$ of these systems we impose the constraint

$$|u_k(t)| \leqslant U_k, \quad t \geqslant 0 \tag{2.10}$$

The values of the constants U_k should be chosen so that the imposed constraint (1.6) is satisfied. From (2.4) and (2.10) we obtain the following estimate

$$|v(x,t)| \leqslant \Sigma U_k |\varphi_k(x)|$$

Consequently, to satisfy the original constraint (1.6), it is sufficient to require that for all $x \in \Omega$ the inequality

$$\Sigma U_k |\varphi_k(x)| \leqslant v^\circ, \quad x \in \Omega \tag{2.11}$$

is satisfied.

We introduce the notation

$$\Phi_k = \max_{x\in\Omega} |\varphi_k(x)| \tag{2.12}$$

Inequality (2.11) is clearly satisfied under the condition

$$\Sigma U_k \Phi_k \leqslant v^0 \tag{2.13}$$

Thus, to solve the control problems for Eqs (1.1) and (1.2), it is sufficient to solve the following control problems for systems (2.5) and (2.6). It is required to construct the feedback controls $u_k(q_k)$ in system (2.5) and $u_k(q_k, \dot{q}_k)$ in system (2.6) for $k = 0, 1, \ldots$, satisfying constraints (2.10) and bringing these systems to the null state [$q_k = 0$ for (2.5) and $q_k = \dot{q}_k = 0$ for (2.6)] in a finite time for any initial conditions of the form (2.8) or (2.9), respectively. Here the constants U_k in (2.10) should satisfy inequality (2.11) for all x, or, which is sufficient, the stronger inequality (2.13).

We note that as a result of applying the Fourier method we have achieved a decomposition of the system: each mode of motion is described by its own Eq. (2.5) or (2.6), with corresponding control u_k. However, the constants U_k in the constraints (2.10) are associated with inequalities (2.11) or (2.13), which is a fundamental difficulty in solving the problem.

For each of Eqs (2.5) and (2.6) we shall construct the time-optimal feedback control u_k under constraint (2.10) for arbitrary fixed U_k. These controls are well known [8]. They are given below together with some additional relations that are necessary for the further analysis of inequalities (2.11), (2.13) and the choice of U_k.

3. FIRST-ORDER EQUATIONS IN TIME

Consider the problem of time-optimal vanishing for one of Eqs (2.5) under constraint (2.10) and initial condition (2.8). We have

$$\begin{gathered} \dot{q}_k + \lambda_k q_k = u_k, \quad |u_k| \leqslant U_k, \quad \lambda_k \geqslant 0 \\ q_k(0) = q_k^0, \quad q_k(T_k) = 0, \quad T_k \to \min \end{gathered} \tag{3.1}$$

The solution of problem (3.1) is elementary. Integrating Eq. (3.1) and satisfying the initial condition, we find that

$$q_k(t) = [q_k^0 + \int_0^t u_k(\tau)\exp(\lambda_k\tau)d\tau]\exp(-\lambda_k t) \tag{3.2}$$

Hence it follows that for the fastest vanishing of the solution $q_k(t)$ the control u_k should be a maximum in modulus and opposite to the sign of the initial value q_k^0, or, equivalently, of the solution $q_k(t)$.

The synthesis of the time-optimal control thus has the form

$$u_k(q_k) = \begin{cases} -U_k \operatorname{sign} q_k, & q_k \neq 0 \\ 0, & q_k = 0 \end{cases} \tag{3.3}$$

The control (3.3) is constant along any phase trajectory. Substituting it into (3.2) and integrating, we obtain

$$q_k(t) = \{|q_k^0| - U_k\lambda_k^{-1}[\exp(\lambda_k t) - 1]\}\exp(-\lambda_k t)\operatorname{sign} q_k^0 \tag{3.4}$$

At the final instant, according to (3.1) we have $q_k(T_k)=0$. From (3.4) we find the instant the process ends

$$T_k=\lambda_k^{-1}\ln(1+\lambda_k|q_k^0|U_k^{-1}),\quad \lambda_k>0,\quad k\geqslant 1$$
$$T_0=|q_0^0|U_0^{-1},\quad \lambda_0=0 \tag{3.5}$$

The solution of the time-optimal control problem (3.1) for all $k\geqslant 0$ is presented in the form of syntheses of the optimal control (3.3). The phase trajectory and optimal time are given by formulae (3.4) and (3.5), respectively.

4. SECOND-ORDER EQUATIONS IN TIME

We will now consider the optimal control problem for one of Eqs (2.6) under constraint (2.10) and initial conditions (2.9). We have

$$\ddot{q_k}+\omega_k^2 q_k=u_k,\quad |u_k|\leqslant U_k,\quad \omega_k\geqslant 0 \tag{4.1}$$
$$q_k(0)=q_k^0,\quad \dot{q_k}(0)=(\dot{q_k})^0,\quad q_k(T_k)=\dot{q_k}(T_k)=0,\quad T_k\to\min$$

We first consider the case when $\omega_k>0$, $k\geqslant 1$. We introduce dimensionless variables and parameters

$$t=\omega_k^{-1}\tau,\quad q_k=U_k\omega_k^{-2}y,\quad \dot{q_k}=U_k\omega_k^{-1}z$$
$$u_k=U_k u,\quad T_k=\omega_k^{-1}T_* \tag{4.2}$$

After transformations (4.2), relations (4.1) acquire the normalized form

$$dy/d\tau=z,\quad dz/d\tau=-y+u,\quad |u|\leqslant 1$$
$$y(0)=y^0,\quad z(0)=z^0,\quad y(T_*)=z(T_*)=0,\quad T_*\to\min \tag{4.3}$$

The solution of the time-optimal problem (4.3) is shown [8]. The optimal control synthesis for problem (4.3) can be put in the form

$$u(y,z)=\operatorname{sign}[\psi(y)-z],\quad \psi\neq 0$$
$$u(y,z)=\operatorname{sign}y=-\operatorname{sign}z,\quad \psi=0 \tag{4.4}$$

The function $\psi(y)$ is given by the equalities

$$\psi(y)=(-y^2-2y)^{1/2},\quad -2\leqslant y\leqslant 0$$
$$\psi(y)=\psi(y+2),\qquad y<-2 \tag{4.5}$$
$$\psi(y)=-\psi(-y),\qquad y>0$$

The switching curve $z=\psi(y)$ given by the relations (4.4) and (4.5) possesses central symmetry and consists of semicircles of unit radii with centres at the points

$$z=0,\quad y=\pm(2i+1),\quad i=0,1,\ldots \tag{4.6}$$

The plus sign in (4.6) gives semicircles in the fourth quadrant of the y, z phase plane, and the minus sign in the second quadrant.

The optimal phase trajectory corresponding to the synthesis of the control (4.4) consists of circular arcs with centres at the points $y=\pm1$, $z=0$. Here, in the domain $z>\psi(y)$, where $u=-1$, the centre of these circles is at the point $y=-1$, $z=0$, while in the domain $z<\psi(y)$, where $u=1$, it is at the point $y=1$, $z=0$. The semicircles of the switching curve with centres at the points $y=\pm1$, $z=0$ are themselves segments of the phase trajectories.

In Fig. 1 the solid lines give the switching curve, and the thin line is one of the optimal trajectories. The arrows show the direction of increasing time.

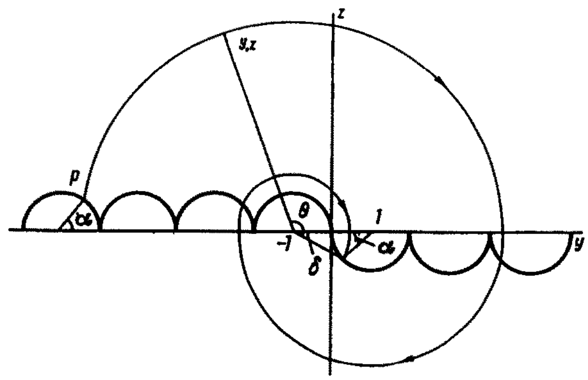

FIG. 1.

We will estimate the time of motion $T_*(y, z)$ along the optimal phase trajectory, beginning at some point y, z. Suppose, to fix our ideas that this point lies in the domain $z > \psi(y)$. We will first make some auxiliary constructions.

We denote by r, θ the polar coordinates of the initial point y, z the pole being the point $y = -1$, $z = 0$. We have

$$y = r\cos\theta - 1, \quad z = r\sin\theta \tag{4.7}$$

The initial segment of the phase trajectory is a circular arc $r = \text{const}$. We continue this arc in an anticlockwise direction until it intersects the switching curve $z = \psi(x, y)$. Suppose the point of intersection P lies on the ith (counting from the origin of coordinates) semicircle of the switching curve (see Fig. 1, where $i = 4$). This means that the coordinate of P can be put in the form

$$\begin{gathered} y_P = -2i + 1 + \cos\alpha, \quad z_P = \sin\alpha \\ i = 2, 3, \ldots, \quad \alpha \in [0, \pi) \end{gathered} \tag{4.8}$$

The angle α corresponds to the arc cut out by the point P from the semicircle of the switching curve on which it lies. We note that such arcs α are cut out by the optimal trajectory from all the semicircles of the switching curve which it intersects. The final arc of the phase trajectory also has angular dimensions α, see Fig. 1.

Since the point P with coordinates (4.8) lies on a circle $r = \text{const}$, we have

$$r^2 = (y_P + 1)^2 + z_P^2 = 4(i-1)^2 + 1 - 4(i-1)\cos\alpha \tag{4.9}$$

We denote by R the length of the radius-vector of the phase point y, z. Using relation (4.7), we obtain

$$R^2 = y^2 + z^2 = (r-1)^2 + 2r(1-\cos\theta) \tag{4.10}$$

The inequalities

$$R \geqslant r - 1 \geqslant [4(i-1)^2 - 4(i-1) + 1]^{1/2} - 1 = 2i - 4 \tag{4.11}$$

follow from (4.10) and (4.9).

The time of motion along any arc of the optimal trajectory can easily be seen to be equal to the angular length of this arc. Each arc between neighbouring switches of the control is either equal to π, or (for the first and second sections) does not exceed π, and the total number of sections is equal to the integer i introduced above. Hence we have $T_* \leqslant \pi i$. Using inequality (4.11) we obtain the estimate

$$T_* \leqslant \pi(R/2 + 2) \equiv T^0(R) \tag{4.12}$$

Estimate (4.12) holds for all $R \geqslant 0$, but it does not imply that $T_* \to 0$ as $R \to 0$. Hence we obtain yet another estimate for sufficiently small R.

Suppose $i=2$, i.e. there is only one switch of the control, see Fig. 1. In this case the optimal trajectory consists of an arc of radius r and angular dimensions $\theta+\delta$ and an arc of radius 1 and angular dimensions α, coinciding with a segment of the switching curve. We denote by δ the angle between the z axis and the ray continued from the point $y=-1$, $z=0$ to the point of the trajectory where the switch occurs. Thus

$$T_* = \theta + \delta + \alpha \tag{4.13}$$

where, as can be determined with the help of Fig. 1, we have

$$\sin\delta = r^{-1}\sin\alpha, \quad \delta \in [0, \pi/2] \tag{4.14}$$

We will obtain some auxiliary relations, which we shall require in order to estimate the time (4.13). Putting $i=2$ in (4.9), we find

$$r = [1 + 8\sin^2(\alpha/2)]^{1/2} \tag{4.15}$$

Equations (4.14) and (4.15) determine the dependence of the angle δ on α. Investigation of this dependence shows that as the angle α varies between the limits in (4.8), the angle δ varies between the limits $[0, \pi/6]$, and $\delta \leqslant \alpha$ always. Thus we have

$$0 \leqslant \delta \leqslant \pi/6, \quad \delta \leqslant \alpha, \quad 0 \leqslant \alpha < \pi \tag{4.16}$$

We note the following inequality

$$\sin(\gamma/2) \geqslant \gamma/\pi, \quad \gamma \in [0, \pi] \tag{4.17}$$

Putting $\gamma=\alpha$ in inequality (4.17), we obtain from (4.15) the relation

$$r \geqslant (1 + 8\pi^{-2}\alpha^2)^{1/2}, \quad \alpha \in [0, \pi)$$

which we rewrite in the form

$$r \geqslant g(\xi) = (1 + \xi)^{1/2}, \quad \xi = 8\pi^{-2}\alpha^2, \quad \xi \in [0, 8) \tag{4.18}$$

Because $g(\xi)$ is a concave function, the inequality

$$[g(\xi) - g(0)]\xi^{-1} \geqslant [g(8) - g(0)]/8, \quad \xi \in [0, 8]$$

is satisfied in the interval under consideration.

Substituting into the last equality the values $g(0)=1$ and $g(8)=3$ from (4.18), we obtain

$$g(\xi) = (1 + \xi)^{1/2} \geqslant 1 + \xi/4, \quad \xi \in [0, 8]$$

which gives the possibility of simplifying relation (4.18)

$$r \geqslant 1 + 2\pi^{-2}\alpha^2, \quad \alpha \in [0, \pi) \tag{4.19}$$

We now transform relation (4.10) using inequality (4.17) for $\gamma=\theta$. We have

$$R^2 = (r-1)^2 + 4r\sin^2(\theta/2) \geqslant (r-1)^2 + 4\pi^{-2}r\theta^2$$

We substitute (4.19) into the latter inequality. We obtain

$$R^2 \geqslant 4\pi^{-4}\alpha^4 + 4\pi^{-2}\theta^2$$

From this the following two inequalities follow

$$R \geqslant 2\pi^{-2}\alpha^2, \quad R \geqslant 2\pi^{-1}|\theta| \tag{4.20}$$

We now transform equality (4.13) for T_*, using inequalities (4.16) and (4.20)

$$T_* = \theta + \delta + \alpha \leqslant 2\alpha + \theta \leqslant 2|\alpha| + |\theta| \leqslant \pi[(2R)^{1/2} + R/2] \equiv T^1(R) \tag{4.21}$$

We compare estimates (4.12) and (4.21). We recall that estimate (4.21) was obtained for $i=2$, and estimate (4.12) for all $i \geqslant 2$. But according to (4.11), for $i \geqslant 3$ we have $R \geqslant 2$. From (4.12) and (4.21) it follows that $T^0(R) \leqslant T^1(R)$ for $R \geqslant 2$. Consequently, for all $i \geqslant 3$ we have $T^0(R) \leqslant T^1(R)$.

It has thus been established that estimate (4.21)

$$T_* \leqslant T^1(R) = \pi[R/2 + (2R)^{1/2}], \quad R = (y^2 + z^2)^{1/2} \tag{4.22}$$

holds for all y, z.

Returning to the original dimensional variables (4.2), we obtain an estimate for the optimal time for problem (4.1) in the form

$$\begin{gathered} T_k(q_k, \dot{q_k}) \leqslant \pi U_k^{-1}[\rho_k/2 + (2U_k\omega_k^{-1}\rho_k)^{1/2}] \\ \rho_k = [\omega_k^2 q_k^2 + (\dot{q_k})^2]^{1/2}; \quad k = 1, 2, \ldots; \quad \omega_k > 0 \end{gathered} \tag{4.23}$$

Here and below the superscript zero on q_k and $\dot{q}_k$ has been omitted.

We consider separately the case with the zero eigenvalue $k = 0$, $\omega_0 = 0$. In this case the optimal control synthesis for problem (4.1) has the form [8]

$$\begin{gathered} u_0(q_0, \dot{q_0}) = U_0 \operatorname{sign}[\psi_0(q_0) - \dot{q_0}], \quad \psi_0 \neq 0 \\ u_0(q_0, \dot{q_0}) = U_0 \operatorname{sign} q_0 = -U_0 \operatorname{sign} \dot{q_0}, \quad \psi_0 = 0 \\ \psi_0(q_0) = -[2U_0|q_0|]^{1/2} \operatorname{sign} q_0, \quad \psi_0(0) = 0 \end{gathered} \tag{4.24}$$

The optimal time is given by the formula

$$\begin{gathered} T_0(q_0, q_0) = U_0^{-1}\{2[(\dot{q_0})^2/2 - U_0 q_0 \sigma]^{1/2} - \dot{q_0}\sigma\} \\ \sigma = \operatorname{sign}[\psi_0(q_0) - \dot{q_0}] \end{gathered}$$

(it is given in this form in, for example, [7]).

Applying the inequality $(a+b)^{1/2} \leqslant |a|^{1/2} + |b|^{1/2}$ to the given relation, we obtain the estimate

$$\tag{4.25}$$

We have thus obtained relations that will be necessary later in the time-optimal control problem (4.1) for all $k \geqslant 0$. The optimal control synthesis $u_k(q_k, \dot{q}_k)$ for $k \geqslant 1$ in dimensional variables is given by relations (4.4) and (4.5), in which it is necessary to substitute the transformation formulae (4.2). In the case when $k = 0$ the synthesis is given by formulae (4.24). The optimal phase trajectory is also well known [8]. For the optimal time, estimate (4.23) has been obtained for $k \geqslant 1$ and (4.25) for $k = 0$.

5. ANALYSIS OF THE CONSTRAINTS AND CONSTRUCTION OF THE CONTROL

The relations obtained in Secs 3 and 4 contain constants U_k and constraints on the control for the kth mode of motion. We choose these constants so as to reduce the total time of the motion, equal to

$$T = \max_k T_k, \quad k \geqslant 0 \quad \text{or} \quad k \geqslant 1 \tag{5.1}$$

while satisfying constraints (2.11) or (2.13). The index k in (2.11), (2.13) and (5.1) takes the values $0, 1, \ldots$ when there is a zero eigenvalue $\lambda_0 = 0$ in problem (2.2) and values $1, 2, \ldots$ when there is none.

Because T_k increases monotonically as U_k increases, and all the U_k occur linearly with positive coefficients in constraints (2.11) and (2.13), it is natural to choose the U_k by an equality requirement on all the T_k: $T_0 = T_1 = \ldots$. This gives the least possible value for T (given constraints (2.11) and (2.13)) in (5.1).

Following the stated proposal, for the first-order equation we put, in accordance with (3.5)

$$T_k = \lambda_k^{-1} \ln(1 + \lambda_k |\dot{q_k}| U_k^{-1}) = T$$

Here T is a constant to be determined.

From this we find

$$U_k = \lambda_k |q_k^0| [\exp(\lambda_k T) - 1]^{-1}, \quad k \geqslant 0 \tag{5.2}$$

Formula (5.2) holds for all $\lambda_k \geqslant 0$. Substituting (5.2) into inequality (2.13), we obtain

$$\Sigma \lambda_k [\exp(\lambda_k T) - 1]^{-1} |q_k^0| \Phi_k \leqslant v^0 \tag{5.3}$$

As we know, under very general assumptions the eigenvalues λ_k and the maxima of the eigenfunctions Φ_k increase no faster than some power of k. The moduli of the Fourier coefficients $|q_k|$ increase less rapidly than k for any bounded initial function $w_0(x)$. Hence, because of the presence of the exponential factor, the series on the left-hand side of inequality (5.3) converges for all $T>0$. As T takes values from 0 to ∞, the sum of the series decreases monotonically from ∞ to 0. Hence there always exists a $T>0$ for which inequality (5.3) is satisfied. Thus the stated control problem for Eq. (1.1) is always solvable by the proposed method. The time T of the process can be chosen from the condition for satisfying inequality (5.3).

We obtain an upper estimate for the time T using the inequality

$$\lambda_k [\exp(\lambda_k T) - 1]^{-1} \leqslant T^{-1} \tag{5.4}$$

It follows from (5.3) and (5.4) that if T is chosen from the condition

$$T = Q_1/v^0, \quad Q_1 = \Sigma |q_k| \Phi_k < \infty \tag{5.5}$$

then inequality (5.3) is clearly satisfied. Consequently, when the series for Q_1 converges the time T can be chosen according to the simple formula (5.5).

We now consider Eq. (1.2) that is of second order in time. In this case, instead of formulae for times T_k one only has the upper estimates (4.23) and (4.25), hence the equality condition on all the T_k cannot be satisfied exactly. Bearing this in mind, and also to simplify the subsequent formulae, we propose to choose U_k in the form

$$\begin{gathered} U_k = c\rho_k, \quad c > 0, \quad k = 1, 2, \ldots \\ U_0 = \max(c_1 |\dot{q}_0|, \ c_2 |q_0|), \quad c_1 > 0, \quad c_2 > 0 \end{gathered} \tag{5.6}$$

Here c, c_1 and c_2 are constants. Substituting U_k from (5.6) into (4.23) we obtain

$$T_k \leqslant \pi[(2c)^{-1} + 2^{1/2}(\omega_k c)^{-1/2}], \quad k = 1, 2, \ldots$$

The last inequality is not violated if all ω_k are replaced by $\omega_1 \leqslant \omega_k$. We obtain the estimate

$$T_k \leqslant \pi[(2c)^{-1} + 2^{1/2}(\omega_1 c)^{-1/2}] \tag{5.7}$$

When substituting expression (5.6) for U_0 into inequality (4.25) we shall distinguish between two cases. In the first case, when $c_1|\dot{q}_0| \geqslant c_2|q_0|$, we obtain from (4.25) and (5.6)

$$T_0 \leqslant (2^{1/2} + 1) c_1^{-1} + 2 |c_1 \dot{q}_0|^{-1/2} |q_0|^{1/2} \leqslant (2^{1/2} + 1) c_1^{-1} + 2 c_2^{-1/2} \tag{5.8}$$

In the second case, when $c_1|\dot{q}_0| < c_2|q_0|$, similar estimates reduce to exactly the same result (5.8). We choose the constants c_1 and c_2 so that both terms on the right-hand sides of inequalities (5.7) and (5.8) are identical term by term, i.e.

819a

From this we find the required constants

$$\begin{gathered} c_1 = \nu_1 c, \quad c_2 = \nu_2 c \\ \nu_1 = 2(2^{1/2} + 1)\pi^{-1} \approx 1{,}53; \quad \nu_2 = 2\omega_1 \pi^{-2} \end{gathered} \tag{5.9}$$

Using (5.9), formulae (5.6) can be written in the form

$$U_k = c\rho_k, \quad k \geqslant 1, \quad U_0 = c\max(\nu_1 |\dot{q}_0|, \ \nu_2 |q_0|) \tag{5.10}$$

The quantities ν_1 and ν_2 are defined in (5.9) and do not depend on c. Because the right-hand sides of inequalities (5.7) and (5.8) are identical by virtue of the choice of constants c_1 and c_2, estimate (5.7) holds for all $k \geqslant 0$. Thus in all cases we have the estimate

$$T \leqslant \pi[(2c)^{-1} + 2^{1/2}(\omega_1 c)^{-1/2}] \tag{5.11}$$

for the time of the control process (5.1).

It remains to choose the constant c so that the constraint (2.11) is satisfied. Substituting (5.10) into (2.11), we obtain

$$c \leqslant v^0[Q^* + \max(\nu_1|\dot{q}_0|,\ \nu_2|q_0|)]^{-1} \tag{5.12}$$

Here we have introduced the notation

$$Q^* = \sup_{x\in\Omega} Q_2(x),\quad Q_2(x) = \Sigma\rho_k|\varphi_k(x)|$$
$$\rho_k = [\omega_k^2 q_k^2 + (\dot{q}_k)^2]^{1/2},\quad k \geqslant 1 \tag{5.13}$$

and used formulae (4.23) for the ρ_k. Inequality (5.13) is written for the case when the zero eigenvalue is present. When it is not present one simply omits the last term (max) in (5.12).

Thus a sufficient condition for the control problem to be solvable for Eq. (1.2) using the proposed approach is uniform boundedness of the series for $Q_2(x)$ from (5.13) in the domain Ω. For this it is sufficient to require the uniform boundedness in Ω of the following two series

$$Q_3(x) = \Sigma\omega_k|q_k||\varphi_k(x)|,\quad Q_4(x) = \Sigma|\dot{q}_k||\varphi_k(x)| \tag{5.14}$$

Using the notation (2.12), the boundedness condition on Q^* from (5.13) can be replaced by the stronger condition of the convergence of the numerical series

$$Q_5 = \Sigma\rho_k\Phi_k < \infty,\quad \rho_k = [\omega_k^2\Phi_k^2 + (\dot{q}_k)^2]^{1/2} \tag{5.15}$$

or the condition of the convergence of the two series

$$Q_6 = \Sigma\omega_k|q_k|\Phi_k < \infty,\quad Q_7 = \Sigma|\dot{q}_k|\Phi_k < \infty \tag{5.16}$$

We will sum up the results obtained. For both equations (1.1) and (1.2) the solvability conditions have been stated and upper limits have been given on the control process time T.

Problem (1.1) is always solvable, and its time T can be chosen from condition (5.3) or, when the series Q_1 converges, from the simpler condition (5.5).

Problem (1.2) is clearly solvable if one of the series convergence conditions (5.13)–(5.16) is satisfied. We have the estimate (5.11) for the time T, in which the constant c should be chosen by condition (5.12). Here the number Q^* is determined from relations (5.13) or one of the following relations

$$Q^* = \sup_{x\in\Omega} Q_3(x) + \sup_{x\in\Omega} Q_4(x),\quad Q^* = Q_5,\quad Q^* = Q_6 + Q_7$$

according to which of the series convergence conditions (5.14)–(5.16) is satisfied.

We remark that when the initial functions w_0 and w_{t0} tend uniformly to zero, all their Fourier coefficients tend to zero, and here all the series in (5.3), (5.5), (5.13)–(5.16) also tend to zero. From estimates (5.3), (5.11), (5.12) it follows that the process time $T \to 0$ for both Eqs (1.1) and (1.2).

After determining the time T and the constant c we find U_k from relations (5.2) and (5.10) for Eqs (1.1) and (1.2), respectively. The coefficients u_k of the required control law (2.4) are found in the form of a synthesis, i.e. depending on the current values q_k and $\dot{q}_k$, in Secs 3 and 4 for Eqs (1.1) and (1.2), respectively, see (3.3) and (4.4). Because the optimal trajectories are known for the systems of Secs 3 and 4, the controls obtained in the form of a synthesis can also be represented in the form of a program $u_k(t)$, i.e. in the form of bang-bang functions with switches points depending on the intial conditions.

Thus the control (2.4) can be represented either in the form of a programmed control for given initial conditions, or in the form of a synthesis, if controls u_k depending on q_k and $\dot{q}_k$ are used. In the second case the control is organized in the form $v = v[x^*;\ w(\cdot, t)]$ for system (1.1) and in the form $v = v[x^{*}; w(\cdot, t), w_t(\cdot, t)]$ for system (1.2). The notation introduced shows that the control v at a point $x \in \Omega$ at time t is a functional of the functions $w(y, t)$ and $w_t(y, t)$ with $y \in \Omega$. However, here the dependence on the initial functions w_0 and w_{0t} is also preserved by means of the constants U_k which depend on the initial data, see (5.2) and (5.10). In these formulae the constants T and c also depend on the initial conditions.

The control (2.4) obtained is by construction such that all boundary and initial conditions together with the constraint (1.6) are satisfied automatically. This control is near to being time-optimal because, firstly, the controls for each subsystem are optimal, and secondly, the bounds U_k are chosen so that the control times for the subsystems are equal or nearly equal to one another.

Below we consider some specific examples in which the convergence conditions for series (5.5), (5.15) and (5.16) are analysed. The conditions for the problems to be solvable are obtained in the form of requirements on the initial functions. In conclusion, some general conditions for the control problem to be solvable for Eq. (1.2) are given.

6. THE ONE-DIMENSIONAL PROBLEM ($n = 1$, $A = \Delta$)

We first consider the heat-conduction and oscillation equations for the case of one spatial variable x. Equations (1.1) and (1.2) have the form

$$w_t = w_{xx} + v, \quad w_{tt} = w_{xx} + v \tag{6.1}$$

The domain Δ is the interval $[0, a]$ of the x axis, and its boundary consists of the two points $x = 0$, $x = a$. We shall consider in tandem conditions (1.3) of Dirichlet and Neumann type

$$w(0) = w(a) = 0, \quad w_x(0) = w_x(a) = 0 \tag{6.2}$$

The eigenfunctions $\varphi_k(x)$ corresponding to problems (6.1) and (6.2) satisfy the equations

$$\varphi_k'' = -\lambda_k \varphi_k, \quad 0 < x < a \tag{6.3}$$

where the primes denote differentiation with respect to x, together with Dirichlet or Neumann conditions

$$\varphi_k(0) = \varphi_k(a) = 0, \quad \varphi_k'(0) = \varphi_k'(a) = 0 \tag{6.4}$$

The eigenvalues of problems (6.3), (6.4) are as follows:

$$\lambda_k = \omega_k^2, \quad \omega_k = \pi k / a \tag{6.5}$$

where $k \geq 1$ for the Dirichlet problem and $k \geq 0$ for the Neumann problem. The orthonormalized eigenfunctions for the Dirichlet and Neumann problems are, respectively, equal to

$$\begin{gathered} \varphi_k(x) = (2/a)^{1/2} \sin(\omega_k x), \quad k = 1, 2, \ldots \\ \varphi_0(x) = a^{-1/2}, \quad \varphi_k(x) = (2/a)^{1/2} \cos(\omega_k x) \end{gathered} \tag{6.6}$$

The quantities Φ_k from (2.12) are bounded in this case

$$\Phi_k = (2/a)^{1/2}, \quad k \geq 1, \quad \Phi_0 = a^{-1/2} \tag{6.7}$$

We shall compute the Fourier coefficients (2.8) and (2.9), assuming that the initial functions $w_0(x)$ and $w_{t0}(x)$ are differentiable with respect to x a sufficient number of times and using integration by parts. With the help of (6.6) we obtain

$$\begin{aligned} q_k(0) &= \int_0^a w_0 \varphi_k dx = \left(\frac{2}{a}\right)^{1/2} \omega_k^{-1} \{ [(-w_0) \cos(\omega_k x)] |_0^a + \int_0^a w_0' \cos(\omega_k x) dx \} = \\ &= \left(\frac{2}{a}\right)^{1/2} \omega_k^{-1} \{ [(-w_0) \cos(\omega_k x)] |_0^a - \omega_k^{-1} \int_0^a w_0'' \sin(\omega_k x) dx \} = \\ &= \left(\frac{2}{a}\right)^{1/2} \omega_k^{-1} \{ [(-w_0 + \omega_k^{-2} w_0'') \cos(\omega_k x)] |_0^a + \omega_k^{-3} \int_0^a w_0^{\mathrm{IV}} \sin(\omega_k x) dx \} \end{aligned} \tag{6.8}$$

for the Dirichlet problem and

$$q_k(0)=\left(\frac{2}{a}\right)^{1/2}\omega_k^{-2}\left\{[w_0'\cos(\omega_k x)]|_0^a+\omega_k^{-1}\int_0^a w_0'''\sin(\omega_k x)\,dx\right\}=$$

$$=\left(\frac{2}{a}\right)^{1/2}\omega_k^{-2}\left\{[(w_0'-\omega_k^{-2}w_0''')\cos(\omega_k x)]|_0^a-\omega_k^{-3}\int_0^a w_0^{\mathrm{V}}\sin(\omega_k x)\,dx\right\},\quad k\geqslant 1 \tag{6.9}$$

for the Neumann problem. From relations (6.8) and (6.9) one can derive estimates for the Fourier coefficients depending, firstly, on the degree of smoothness of the initial function w_0 and secondly, on additional conditions at the boundary points $x = 0$ and $x = a$, i.e. on Γ. We drop the argument 0 of the function q_k. Henceforth the B_j are some positive constants and the C^i are classes of functions having continuous derivatives in the interval $[0, a]$ up to order i inclusive. For the Dirichlet problem we obtain, using (6.8),

$$\begin{aligned}
|q_k| &\leqslant B_1\omega_k^{-1} && \text{for} \quad w_0\in C^1\\
|q_k| &\leqslant B_2\omega_k^{-2} && \text{for} \quad w_0\in C^2,\quad w_0=0 \text{ on } \Gamma\\
|q_k| &\leqslant B_3\omega_k^{-3} && \text{for} \quad w_0\in C^3,\quad w_0=0 \text{ on } \Gamma\\
|q_k| &\leqslant B_4\omega_k^{-4} && \text{for} \quad w_0\in C^4,\quad w_0=w_0''=0 \text{ on } \Gamma
\end{aligned} \tag{6.10}$$

For the Neumann problem we similarly have

$$\begin{aligned}
|q_k| &\leqslant B_5\omega_k^{-1} && \text{for} \quad w_0\in C^1\\
|q_k| &\leqslant B_6\omega_k^{-2} && \text{for} \quad w_0\in C^2\\
|q_k| &\leqslant B_7\omega_k^{-3} && \text{for} \quad w_0\in C^3,\quad w_0'=0 \text{ on } \Gamma\\
|q_k| &\leqslant B_8\omega_k^{-4} && \text{for} \quad w_0\in C^4,\quad w_0'=0 \text{ on } \Gamma
\end{aligned} \tag{6.11}$$

Obviously, estimates of the form (6.10) and (6.11) can be continued without limit. For the Fourier coefficients $q_k(0)$ from (2.9) we have estimates similar to (6.10) and (6.11), with w_0 replaced by w_{t0}.

Turning to the investigation of the convergence of the series in (5.5) and (5.6), we note that according to (6.7) the quantities Φ_k are independent of k. Using also relation (6.5), we obtain the following convergence conditions for the series.

Series (5.5) for the Dirichlet problem converges under the conditions

$$w_0\in C^2,\quad w_0=0 \text{ on } \Gamma \tag{6.12}$$

and for the Neumann problem under the condition

$$w_0\in C^2 \tag{6.13}$$

The series (5.16) for the Dirichlet problem converges under the conditions

$$w_0\in C^3,\quad w_{t0}\in C^2,\quad w_0=w_{t0}=0 \text{ on } \Gamma \tag{6.14}$$

and for the Neumann problem under the conditions

$$w_0\in C^3,\quad w_{t0}\in C^2,\quad \partial w_0/\partial n=0 \text{ on } \Gamma \tag{6.15}$$

We note that convergence conditions (6.12) and (6.14) for series (5.5) and (5.16) for the Dirichlet problem include, as well as smoothness requirements, Dirichlet conditions on the initial functions w_0 and w_{t0}. Generally speaking, such conditions are not necessary in the statement of initial-boundary-value problems, and they are an additional imposition. In the case of the Neumann problem, however, conditions (6.13) and (6.15) are less restrictive: for series (5.5) no conditions other than smoothness are imposed, while for series (5.16) the Neumann condition is only imposed on the initial function w_0 (and not on the function w_{t0}).

We recall that the control problem for the first equation of (6.1) (the heat conduction equation) is always solvable, and conditions (6.12) and (6.13) ensuring the convergence of series (5.5) are there only to apply the simple estimate of the control process time in (5.5). For the second equation of (6.1) (the vibrating string equation) conditions (6.14) and (6.15) are sufficient conditions for the control problem to be solvable by the proposed methods.

7. CONTROL OF BEAM OSCILLATIONS ($n = 1$, $A = \Delta^2$)

As an example of a fourth-order equation we consider the control of transverse oscillations of an elastic beam. Equation (1.2) in this case has the form

$$w_{tt}=-w_{xxxx}+v \tag{7.1}$$

To fix our ideas we will restrict ourselves to hinged support boundary conditions at both ends of a beam of length a, i.e.

$$w = w_{xx} = 0 \text{ on } \Gamma, \quad \Gamma = \{ x = 0, \ x = a \} \tag{7.2}$$

The eigenvalue problem (2.1) for system (7.1), (7.2) has the form

$$\varphi^{\mathrm{IV}} = \lambda\varphi, \quad x \in \Omega = [0, a], \quad \varphi = \varphi'' = 0 \text{ on } \Gamma \tag{7.3}$$

It is well known that the eigenvalues of problem (7.3) are positive and are

$$\lambda_k = \omega_k^2, \quad \omega_k = (k\pi/a)^2, \quad k = 1, 2, \ldots \tag{7.4}$$

where the ω_k are interpreted as the frequencies of the natural oscillations of the beam. The corresponding eigenfunctions of problem (7.3) can be represented in the form of equalities (6.6). Hence estimates (6.7), (6.8) and (6.10) remain valid for the problem under consideration, but throughout (6.6), (6.8) and (6.10) the frequencies ω_k are now defined by formulae (7.4) [instead of (6.5)]. Using the given estimates, we obtain like (6.14), the following sufficient conditions for series (5.16) to converge in the problem under consideration:

$$w_0 \in C^2, \quad w_{t0} \in C^1, \quad w_0 = 0 \text{ on } \Gamma \tag{7.5}$$

Conditions (7.5) include only one of the two boundary conditions (7.2) on Γ. They are less restrictive than (6.14) and are certainly satisfied under those restrictions which are normally imposed on the initial functions in the beam oscillation problem.

8. THE TWO-DIMENSIONAL AND THREE-DIMENSIONAL PROBLEMS ($n = 2, 3$; $A = \Delta$)

We now consider the equations

$$w_t = \Delta w + v, \quad w_{tt} = \Delta w + v; \quad n = 2, 3 \tag{8.1}$$

in the two-dimensional and three-dimensional cases. Syppose the domain Ω is a rectangle when $n = 2$ and a rectangular parallelepiped when $n = 3$, i.e. specified by

$$\Omega: \ 0 < x_i < a_i; \quad i = 1, \ldots, n; \quad n = 2, 3 \tag{8.2}$$

The solutions of the eigenvalue problem (2.2) for Eqs (8.1) in domains (8.2) under Neumann and Dirichlet conditions are known and are obtained by separation of variables. In the two-dimensional ($n = 2$) Dirichlet case we obtain, like (6.5) and (6.6)

$$\begin{aligned} &\lambda_{ik} = \omega_{ik}^2 = \pi^2 [(i/a_1)^2 + (k/a_2)^2]; \quad i, k = 1, 2, \ldots \\ &\varphi_{ik}(x_1, x_2) = 2(a_1 a_2)^{-1/2} \sin(\pi i x_1/a_1) \sin(\pi k x_2/a_2) \end{aligned} \tag{8.3}$$

For the Neumann problem the eigenvalues are given by relations (8.3) for $i, k = 0, 1, \ldots$, while the eigenfunctions have a form similar to (6.6)

$$\begin{aligned} &\varphi_{ik}(x_1, x_2) = 2(a_1 a_2)^{-1/2} \cos(\pi i x_1/a_1) \cos(\pi k x_2/a_2) \\ &\varphi_{00}(x_1, x_2) = (a_1 a_2)^{-1/2} \\ &\varphi_{0k} = 2^{1/2}(a_1 a_2)^{-1/2} \cos(\pi k x_2/a_2) \\ &\varphi_{i0} = 2^{1/2}(a_1 a_2)^{-1/2} \cos(\pi i x_1/a_1); \quad i, k = 1, 2, \ldots \end{aligned} \tag{8.4}$$

By (8.3) and (8.4) the quantities (2.12) are bounded

$$\Phi_{ik} = 2(a_1 a_2)^{-1/2}; \quad i, k = 1, 2, \ldots \tag{8.5}$$

We will now estimate the Fourier coefficients (2.8) and (2.9), assuming that the initial functions w_0 and w_{t0} are sufficiently smooth. Replacing the multiple integrals over the domain Ω by repeated integration over x_1, x_2, and then using integration by parts, we obtain, like (6.8)–(6.11), the following estimates

$$\begin{aligned} &|q_{ik}| \leq B_1 (ik)^{-1} \quad \text{for} \quad w_0 \in C^{(1)} \\ &|q_{ik}| \leq B_2 (ik)^{-2} \quad \text{for} \quad w_0 \in C^{(2)}, \quad w_0 = 0 \text{ on } \Gamma \\ &|q_{ik}| \leq B_3 (ik)^{-3} \quad \text{vor} \quad w_0 \in C^{(3)}, \quad w_0 = 0 \text{ on } \Gamma \end{aligned} \tag{8.6}$$

for the Dirichlet problem and

$$
\begin{aligned}
&|q_{ik}| \leq B_4 (ik)^{-1}, \quad |q_{0k}| \leq B_5 k^{-1} \\
&|q_{i0}| \leq B_6 i^{-1} \quad \text{for} \quad w_0 \in C^{(1)} \\
&|q_{ik}| \leq B_7 (ik)^{-2}, \quad |q_{0k}| \leq B_8 k^{-2} \\
&|q_{i0}| \leq B_9 i^{-2} \quad \text{for} \quad w_0 \in C^{(2)} \\
&|q_{ik}| \leq B_{10} (ik)^{-3}, \quad |q_{0k}| \leq B_{11} k^{-3} \\
&|q_{i0}| \leq B_{12} i^{-3} \quad \text{for} \quad w_0 \in C^{(3)}, \quad \partial w_0/\partial n = 0 \text{ on } \Gamma
\end{aligned}
\tag{8.7}
$$

for the Neumann problem. In (8.6) and (8.7) $i, k = 1, 2, \ldots$, everywhere, while $C^{(r)}$ is the class of functions w having continuous partial derivatives of the form

$$
\partial^{p+q}/\partial x_1^p \partial x_2^q, \quad 0 \leq p \leq r, \quad 0 \leq q \leq r \tag{8.8}
$$

in the closed domain Ω.

For the Fourier coefficients $q^{\bullet}_{ik}(0)$ from (2.9) there are estimates similar to (8.6) and (8.7), with w_0 replaced by w_{t0}.

Using relations (8.3), (8.5)–(8.7) we obtain the required sufficient conditions for series (5.5) and (5.16) to converge. In the cases considered here summation in these series is performed over two indices i and k, from 1 to ∞ for the Dirichlet problem and from 0 to ∞ for the Neumann problem.

It turns out that series (5.5) converges for the Dirichlet problem under the conditions

$$
w_0 \in C^{(2)}, \quad w_0 = 0 \text{ on } \Gamma \tag{8.9}
$$

and for the Neumann problem under the condition

$$
w_0 \in C^{(2)} \tag{8.10}
$$

Series (5.16) converge for the Dirichlet problem under the conditions

$$
w_0 \in C^{(3)}, \quad w_{t0} \in C^{(2)}, \quad w_0 = w_{t0} = 0 \text{ on } \Gamma \tag{8.11}
$$

and for the Neumann problem under the conditions

$$
w_0 \in C^{(3)}, \quad w_{t0} \in C^{(2)}, \quad \partial w_0/\partial n = 0 \text{ on } \Gamma \tag{8.12}
$$

The convergence conditions (8.9)–(8.12) are completely analogous to the corresponding conditions (6.12)–(6.15) for the one-dimensional problem.

In the three-dimensional case ($n = 3$), which is completely analogous to the two-dimensional one, the eigenvalues are given by equalities similar to (8.3)

$$
\lambda_{ijk} = \pi^2 [(i/a_1)^2 + (j/a_2)^2 + (k/a_3)^2]
$$

Here $i, j, k \geq 1$ for the Dirichlet problem and $i, j, k \geq 0$ for the Neumann problem.

Formulae and estimates similar to (8.3)–(8.5) hold for the eigenfunctions and Fourier coefficients. Finally, we arrive at exactly the same convergence conditions (8.9)–(8.12) as in the two-dimensional case. Here, as in (8.8), $C^{(r)}$ is the class of functions w having continuous partial derivatives of the form

$$
\partial^{p+q+s}/\partial x_1^p \partial x_2^q \partial x_3^s, \quad 0 \leq p \leq r, \quad 0 \leq q \leq r, \quad 0 \leq s \leq r
$$

in the closed domain Ω.

9. THE SOLUBILITY CONDITIONS IN THE GENERAL CASE

As was pointed out in Sec. 5, no additional conditions are required for the control problem to be solvable for Eq. (1.1), while for the control of (1.2) it is sufficient, for example, that the functions $Q_3(x)$ and $Q_4(x)$ from (5.14) be uniformly bounded in Ω. We shall analyse these conditions.

Below we shall always assume sufficient smoothness of the coefficients of the operators of A from (1.2) and M from (1.3), and also of the boundaries Γ and initial functions w_0 and w_{t0} from (1.5).

We note that the series (5.14) contain, firstly, eigenfunctions $\varphi_k(x)$ of problem (2.2), and secondly, Fourier coefficients q_k and $q^{\bullet}_k$ of the initial functions w_0 and w_{t0}. It is therefore desirable to use the following estimates for the series (5.14), which follow from the Cauchy inequality and enable us to separate the contributions of the eigenfunctions and Fourier coefficients

$$
\begin{aligned}
Q_3(x) &\leq [\Sigma \lambda_k^{-\beta} \varphi_k^2(x) \cdot \Sigma \lambda_k^{1+\beta} q_k^2]^{1/2} \\
Q_4(x) &\leq [\Sigma \lambda_k^{-\gamma} \varphi_k^2(x) \cdot \Sigma \lambda_k^{\gamma} (q^{\bullet}_k)^2]^{1/2}
\end{aligned}
\tag{9.1}
$$

Here β and γ are currently arbitrary numbers, which will be chosen later so that all the series in (9.1) are bounded.

We shall consider fractional (positive and negative) powers of the differential operator A. An operator A of order $2m$ defines a transformation $Aw = f$. Its domain of definition D_A is the class of functions w defined in the domain Ω having square-integrable partial derivatives up to order $2m$ inclusive (this fact can be expressed in the form $D_A \subset H_{2m}(\Omega)$, where H_{2m} is the corresponding Sobolev space), and also satisfying boundary conditions (1.3).

According to Agmon's kernel theorem [9], for $2ms > n$ the operator A^{-s} is an integral operator with a continuous kernel equal to

$$K(x, y) = \Sigma \lambda_k^{-s} \varphi_k(x) \varphi_k(y)$$

Putting $x = y$, i.e. considering the kernel on the diagonal, we obtain the uniform boundedness of the series

$$\Sigma \lambda_k^{-s} \varphi_k^2(x) \leqslant \text{const} < \infty, \quad 2ms > n$$

It follows from this that for uniform boundedness of the first factors on the right-hand sides of (9.1), i.e. the series depending on x, it is sufficient that

$$\beta > n(2m)^{-1}, \quad \gamma > n(2m)^{-1} \tag{9.2}$$

We remark that conditions (9.2) for $m = 1$ were first given by Il'in [10].

The second factors in the right-hand sides of (9.1) (series depending on the Fourier coefficients) can, by Parseval's equality, be represented in the form

$$\begin{aligned} \Sigma \lambda_k^{1+\beta} q_k^2 &= \int_\Omega (A^{(1+\beta)/2} w_0)^2 dx \\ \Sigma \lambda_k^{\gamma} (\dot{q}_k)^2 &= \int_\Omega (A^{\gamma/2} w_{t0})^2 dx \end{aligned} \tag{9.3}$$

Series (9.3) converge if the functions $A^{(1+\beta)/2} w_0$ and $A^{\gamma/2} w_{t0}$ are square integrable in the domain Ω, i.e. belong to the class $L^2(\Omega)$. In other words, the functions w_0 and w_{t0} should belong to the domains of definition of the corresponding operator, i.e.

$$w_0 \in D_{A^{(1+\beta)/2}}, \quad w_{t0} \in D_{A^{\gamma/2}} \tag{9.4}$$

It follows from Seeley's work [11] that the domain of definition D_{A^s} for $s \in (0, 1)$ lies in $H_{2ms}(\Omega)$ and is distinguished by those boundary conditions (1.3) whose order $\operatorname{ord} M_j = r_j < r = 2ms - ½$. In the case when for some j we have $r_j = r$, the corresponding boundary condition is to be understood in some integral sense.

From (9.4) we have, in the case under consideration

$$\begin{aligned} &s = (1 + \beta)/2, \quad r = m(1 + \beta) - ½ \quad \text{for} \quad w_0 \\ &s = \gamma/2, \quad r = m\gamma - ½ \quad \text{for} \quad w_{t0} \end{aligned} \tag{9.5}$$

where s can also be greater than unity.

Suppose, for example, $s = 1 + \sigma$, where $\sigma \in (0, 1)$. Then, representing the result of the action of the operator A^s in the form $A^s w = A^\sigma(Aw)$ and applying Seeley's theorem, we arrive at the following assertion. The domain of definition D_{A^s} lies in $H_{2ms}(\Omega)$ and is distinguished by boundary conditions (1.3) and also those boundary conditions $M_j A w = 0$ for which $\operatorname{ord} M_j < 2m\sigma - ½$. In other words, for $s \in (1, 2)$, as well as the boundary conditions (1.3), conditions of the form $M_j A w = 0$ for which $\operatorname{ord}(M_j A) < r = 2ms - ½$ are also imposed on the function w.

Similar results also follow from lemmas derived in Appendix 2 of [12].

Thus, for the convergence of series (9.3) the functions w_0 and w_{t0} should satisfy conditions depending on parameters s and r, the stringency of these conditions increasing with s and r. We note

TABLE 1

n, m	$\nu^*(w_0)$	$\nu^*(w_{t0})$	$r^*(w_0)$	$r^*(w_{t0})$
1,1	3/2	1/2	1	0
1,2	5/2	1/2	2	0
2,1	2	1	1	0
2,2	3	1	2	0
3,1	5/2	3/2	2	1
3,2	7/2	3/2	3	1

that for restrictions $r_j < r$ on operator orders the r_j are whole numbers, hence the fractional part of r is not significant.

We determine two numbers for each of the functions w_0 and w_{t0} with the help of relations (9.2) and (9.5): the lower bound s^* on the possible values of s and the integer part r^* of the lower bound on possible values of r. The values of $\nu^* = 2ms^*$ and r^* for various pairs n, m for $n \leqslant 3$, $m \leqslant 2$ are shown in Table 1.

Using the values of ν^* and r^* obtained one can answer the question of the convergence of series (9.1) and thereby obtain sufficient conditions for the control problems under consideration to be solvable.

For this it is sufficient to require that the following conditions be satisfied.

Firstly, the functions w_0 and w_{t0} should belong to classes $H_\nu(\Omega)$, where ν is any number greater than the corresponding ν^*. In particular, ν can be chosen to be an integer, and this requirement will then indicate the existence for the functions w_0 and w_{t0} of square-integrable partial derivatives up to order ν inclusive.

Secondly, the functions w_0 and w_{t0} should satisfy those boundary conditions (9.2) on Γ for which $\operatorname{ord} M_j \leqslant r^*$, and those of the boundary conditions $M_j A w = 0$ for which $\operatorname{ord}(M_j A) \leqslant r^*$. Because $\operatorname{ord} M_j < \operatorname{ord} A = 2m$, the imposition of the conditions $M_j A w = 0$ is only required when $r^* \geqslant 2m$.

It is clear from Table 1 that the inequality $r^* \geqslant 2m$ only holds when $n = 3$, $m = 1$ for the function w_0. In this case for the Dirichlet problem ($\operatorname{ord} M = 0$) we have $\operatorname{ord} MA = 2 = r^*(w_0)$, and it is necessary to impose on w_0 the additional condition $Aw = 0$ on Γ. In the case of the Neumann problem ($\operatorname{ord} M = 1$) for $n = 3$, $m = 1$, and also for all problems with other values of n, m, additional conditions do not appear.

The appearance of an additional boundary condition can be explained as follows. The proposed control law (2.4) vanishes on Γ in the case of the Dirichlet problem because here $\varphi_k = 0$ on Γ. This reduces the possibility of control on the boundary of the domain, and can require additional conditions on the initial functions on Γ.

At the same time some of the boundary conditions (1.3) for the problem to be solvable need not be applied. For example, for $n = 2$, $m = 1$ we have $r^*(w_0) = 1$, $r^*(w_{t0}) = 0$. Consequently, for a second-order operator A in the case of the Dirichlet problem ($\operatorname{ord} M = 0$) the functions w_0 and w_{t0} should satisfy the Dirichlet condition, while in the case of the Neumann problem ($\operatorname{ord} M = 1$) the function w_0 should satisfy the Neumann condition, while the function w_{t0} need not satisfy it.

Comparing the data in the table with the results of the examples in Secs 6–8, we see that in the examples the convergence conditions turned out to be less restrictive for $n = 1$, $m = 2$ and $n = 3$, $m = 1$. For $n = 1$, $m = 2$ in the example it is not required to impose the condition $w_0'' = 0$ on Γ, which figures in the table: $r^*(w_0) = 2$. For $n = 3$, $m = 1$ in the Neumann problem example the condition $\partial w_{t0}/\partial n = 0$ is not required, while for the Dirichlet problem the condition is $\Delta w_0 = 0$ on Γ, as follows from Table 1.

The author expresses his deep thanks to M. S. Agranovich, V. A. Il'in, A. I. Ovseyevich and A. S. Shamayev for valuable advice and discussions.

REFERENCES

1. BUTKOVSKII A. G., *Optimal Control Theory for Systems with Distributed Parameters*. Nauka, Moscow, 1965.
2. LIONS J.-L., *Optimal Control of Systems Described by Partial Differential Equations*. Mir, Moscow, 1972.

3. SIRAZETDINOV T. K., *Optimization of Systems with Distributed Parameters*. Nauka, Moscow, 1977.
4. YEGOROV A. I., *Optimal Control of Thermal and Diffusive Processes*. Nauka, Moscow, 1978.
5. LIONS J.-L., Exact controllability, stabilization and perturbations for distributed systems. *SIAM Review* **30**, 1–68, 1988.
6. UTKIN V. I. and ORLOV Yu. V., *The Theory of Infinitely-dimensional Control Systems in Slipping Regimes*. Nauka, Moscow, 1990.
7. CHERNOUS'KO F. L., Decomposition and suboptimal control for dynamical systems. *Prikl. Mat. Mekh.* **54**, 883–893, 1990.
8. PONTRYAGIN L. S., BOLTYANSKII V. G., GAMKRELIDZE R. V. and MISHCHENKO Ye. F., *The Mathematical Theory of Optimal Processes*. Nauka, Moscow, 1976.
9. AGMON S., On kernels, eigenvalues and eigenfunctions of operators related to elliptic problems. *Communs Pure Appl. Maths.* **18**, 627–663, 1965.
10. IL'IN V. A., On the uniform convergence of eigenfunction expansions throughout a closed domain. *Mat. Sbor.* **45** (87), 195–232, 1958.
11. SEELEY R., Interpolation in L^p with boundary conditions. *Stud. Math.* **44**, 46–60, 1972.
12. IL'IN V. A., *Spectral Theory of Differential Operators*. Nauka, Moscow, 1991.

Translated by R.L.Z.